Received: *August 18, 1999*, Accepted: *September 14, 1999*
Replaced with revised version: *September 29, 1999*

Non-BPS Dirac-Born-Infeld solitons

Theodora Ioannidou

Institute of Mathematics, University of Kent at Canterbury
Canterbury, CT2 7NZ, U.K.
E-mail: T.Ioannidou@ukc.ac.uk

George Papadopoulos

Department of Mathematics, King's College London
Strand, London, NW10 2DU, U.K.
Email: george.papadopoulos@kcl.ac.uk

Paul M. Sutcliffe

Institute of Mathematics, University of Kent at Canterbury
Canterbury, CT2 7NZ, U.K.
E-mail: P.M.Sutcliffe@ukc.ac.uk

Abstract: We show that $\mathbb{CP}^n$ sigma model solitons solve the field equations of a Dirac-Born-Infeld (DBI) action and, furthermore, we prove that the non-BPS soliton/anti-soliton solutions of the sigma model also solve the DBI equations. Using the moduli space approximation we compare the dynamics of the BPS sigma model solitons with that of the associated DBI solitons. We find that for the $\mathbb{CP}^1$ case the metric on the moduli space of sigma model solitons is identical to that of the moduli space of DBI solitons, but for $\mathbb{CP}^n$ with $n > 1$ we show that the two metrics are not equal. We also consider the possibility of similar non-BPS solitons in other DBI theories.

Keywords: Solitons Monopoles and Instantons, D-branes.

JHEP09(1999)016

Contents

1. Introduction

The low energy dynamics of branes in strings and M-theory can be described by actions of Dirac-Born-Infeld (DBI) type [1]. The various classical solutions of these actions admit a geometric interpretation in terms of brane configurations. In fact most of these solutions have a bulk interpretation as intersecting branes [2]. The part of the DBI D-brane action which is quadratic in the derivatives of the fields can be identified with a Yang-Mills theory coupled to matter. This has led to a relation between intersecting brane configurations and classical solutions of Yang-Mills theories. One of the applications of this relation is that some classical and quantum properties of Yang-Mills theory can be understood as geometric properties of brane configurations [3].

The fields of DBI type actions are scalars, vectors or tensors. In certain cases, there is a consistent truncation of the DBI action for which the vector and tensor fields are not active.[1] The remaining fields are scalars and the action reduces to a Dirac one (or Nambu-Goto for the string case). Then powerful geometrical methods can be used to investigate static configurations, like those of (generalized) calibrations [4, 5]. In particular, bounds for the energy can be established and the solutions which saturate them can be found. The part of this reduced DBI action which is quadratic in the

[1] A field is active in a solution, if it has a non-trivial dependence on the brane worldvolume coordinates.

derivatives of the scalars is that of a non-linear sigma model. So the possibility arises to interpret various sigma model solitons in terms of brane configurations and to relate the calibration bounds to sigma model ones. An example was presented in [6, 7], where the sigma model lumps and Q-kinks were understood in terms of M-2-brane configurations. In particular it was observed that the sigma model energy bound associated with lumps on a Taub-Nut target space is related to the Kähler calibration bound in the effective theory of the M-2-brane in the presence of a Kaluza-Klein (KK)-monopole. Moreover, it was found that the solutions that saturate the sigma model bound also solve the corresponding bound of DBI. In this way the sigma model lumps were embedded in the effective theory of the M-2-brane.

It is expected that other sigma model solitons can be embedded in a brane theory, although there are several restrictions on the sigma model target space required by the consistency of the background due to kappa symmetry. Furthermore, for a DBI action it is far from clear that solutions of the sigma model field equations will also solve the DBI field equations since the former is just an approximation of the latter. However, this appears to be the case for a large class of solutions that saturate certain sigma model energy bounds. This is probably due to supersymmetry which protects them against higher derivative corrections. However, as we shall demonstrate, there are sigma model solutions that do not saturate a bound which are also solutions of DBI equations without any modification.

A large class of $(2+1)$-dimensional sigma models are those with target spaces the complex projective spaces $\mathbb{CP}^n$. Since $\mathbb{CP}^n$ is a Kähler manifold, these sigma models admit a supersymmetric extension with $N = 2$ supersymmetry in three dimensions ($N = 1$ supersymmetry in four dimensions) and they are described by chiral superfields. The energy of the static sigma model configurations is bounded by the topological central charge of the supersymmetry algebra. The configurations that saturate this bound are solitons which can be thought of as holomorphic curves in $\mathbb{CP}^n$. Many of the properties of these solitons have been extensively investigated, including their low energy scattering [8] as well as soliton/anti-soliton annihilation using numerical and other methods [9]. Another novel class of solutions of $\mathbb{CP}^n$ sigma models are those describing unstable soliton/anti-soliton states, which of course do not saturate the above bound [10, 11].

In this paper, we consider a DBI action with target space $\mathbb{CP}^n$. Such a DBI action admits an energy bound associated with a degree two Kähler calibration. The part of the DBI action which is quadratic in the derivatives of the fields can be identified with the $\mathbb{CP}^n$ sigma model. The calibrated submanifolds that saturate the bound are holomorphic submanifolds of $\mathbb{R}^2 \times \mathbb{CP}^n$ and these are precisely the static solitons of the (2+1)-dimensional $\mathbb{CP}^n$ sigma model. In addition, we shall show that the soliton/anti-soliton solutions of the $\mathbb{CP}^n$ sigma model are also solutions of the DBI action. These solutions saturate neither the sigma model nor the Kähler calibration bounds.

JHEP09(1999)016

Using the moduli space approximation, we compare the low energy dynamics of BPS sigma model solitons with that of the associated DBI solitons and we find that they may differ. Finally, we investigate the possibility of other non-BPS solitons in DBI theories.

2. The $\mathbb{CP}^n$ sigma model

In this section we briefly review some properties of the (2+1)-dimensional sigma model with target space $\mathbb{CP}^n$ (for more details see eg. [11]). For this, we begin with the description a (2+1)-dimensional sigma model with target space any Kähler manifold M equipped with a metric h and compatible complex structure K. The sigma model fields are maps X from $\mathbb{R}^{(1,2)}$ into M. The energy of a static sigma model configuration ($X : \mathbb{R}^2 \to M$) is

$$E_\sigma = \frac{1}{2} \int d^2x h_{IJ} \delta^{ij} \partial_i X^I \partial_j X^J \tag{2.1}$$

where $i, j = 1, 2$ and $I, J = 1, \dots, \mathrm{dim}M$. Then

$$\begin{aligned} E_\sigma = & \frac{1}{4} \int d^2x\, h_{IJ}\, \delta^{ij} \,(\partial_i X^I \pm \epsilon^k{}_i K^I{}_L \partial_k X^L)(\partial_j X^J \pm \epsilon^\ell{}_j K^J{}_M \partial_\ell X^M) \mp \\ & \mp \frac{1}{2} \int d^2x \,(\omega_K)_{IJ} \epsilon^{ij} \partial_i X^I \partial_j X^J \,, \end{aligned} \tag{2.2}$$

where ϵ is the Levi-Civita tensor and ω_K is the Kähler form of K. So we find

$$E_\sigma \geq 2\pi |Q| \tag{2.3}$$

where

$$Q = \frac{1}{4\pi} \int d^2x (\omega_K)_{IJ} \epsilon^{ij} \partial_i X^I \partial_j X^J \tag{2.4}$$

is the topological charge since ω_K is a closed form. Clearly, the configurations that saturate the bound satisfy

$$\partial_i X^I \pm \epsilon^k{}_i K^I{}_J \partial_k X^J = 0 \,. \tag{2.5}$$

This is the Cauchy-Riemann equation and the solutions are holomorphic curves in the Kähler manifold M.

For $M = \mathbb{CP}^n$, the sigma model fields X can be parameterized by a complex $(n+1)$-component column vector U, of unit length i.e. $U^\dagger U = 1$. The sigma model lagrangean is given by

$$\mathcal{L}_\sigma = (D_\mu U)^\dagger (D^\mu U) \,, \tag{2.6}$$

where $D_\mu V = \partial_\mu V - (U^\dagger \partial_\mu U) V$. Greek indices run over the spacetime values 0,1,2 and we use the Minkowski metric $\eta_{\mu\nu} = (-1, 1, 1)$. The sigma model equation of motion which follows from (2.6) is

$$D_\mu D^\mu U + (D_\mu U)^\dagger (D^\mu U) U = 0 \tag{2.7}$$

and all finite energy solutions are classified by the integer valued topological charge (2.4), which in this parameterization is given by

$$Q = \frac{i}{2\pi} \int d^2x \, \epsilon^{jk} (D_j U)^\dagger (D_k U) \, . \tag{2.8}$$

The configurations that saturate the bound satisfy

$$D_j U = \mp i \epsilon^k{}_j D_k U \, . \tag{2.9}$$

The BPS soliton solutions are easily constructed as

$$U = \frac{f}{|f|} \, , \tag{2.10}$$

where f is any vector whose components are rational functions of the complex coordinate $z = x_1 + ix_2$. The topological charge Q is positive in this case and is equal to the highest degree of the rational functions which occur in the entries of f. The anti-soliton solutions, which have negative topological charge, are obtained in the same way but with f an anti-holomorphic function of z.

There are also non-BPS solutions, i.e. solutions of the second order equations (2.7) which are not solutions of the first order equations (2.9), and therefore do not saturate the energy bound [10, 11]. All these solutions can be obtained explicitly by making use of the operator P_+ acting on a general vector g as

$$P_+ g = \partial_z g - (g^\dagger \partial_z g) \frac{g}{|g|^2} \, . \tag{2.11}$$

The non-BPS solutions are then given by a repeated application of this operator

$$U = \frac{P_+^k f}{|P_+^k f|} \, , \qquad k = 0, \ldots, n \tag{2.12}$$

starting with any initial holomorphic vector f. The operator P_+ can only be applied n times, since after this number of applications the original holomorphic vector is converted into an anti-holomorphic vector and P_+ annihilates anti-holomorphic vectors. Thus this operator converts BPS solitons into BPS anti-solitons, but intermediate solutions with $0 < k < n$ correspond to non-BPS soliton/anti-soliton configurations. Note that in the case of the $\mathbb{CP}^1$ model the operator can be applied only once and converts BPS soliton solutions to BPS anti-soliton solutions, so there are no non-BPS solutions in this case.

3. The $\mathbb{CP}^n$ DBI model

To define the $\mathbb{CP}^n$ DBI model we begin with the spacetime $\mathbb{R}^{(1,2)} \times \mathbb{CP}^n$ with metric

$$d\tilde{s}^2 = ds^2(\mathbb{R}^{(1,2)}) + ds^2(\mathbb{CP}^n) \, , \tag{3.1}$$

where $ds^2(\mathbb{CP}^n)$ is the Fubini-Study metric on $\mathbb{CP}^n$. We then consider a (2+1)-

dimensional embedded submanifold on which the action is obtained as the world-volume of the induced metric obtained by pulling back the spacetime metric (3.1) to the embedded submanifold. Explicitly,

$$I_{DBI} = \int d^3x \sqrt{\det(\tilde{\gamma}_{\mu\nu})} \tag{3.2}$$

where $\tilde{\gamma}_{\mu\nu}$ is the induced metric i.e. the pull back of $d\tilde{s}^2$ to the submanifold. We choose the static gauge, in which the spacetime coordinates of the submanifold are identified with the $\mathbb{R}^{(1,2)}$ part of the spacetime metric, and then clearly the part of the action (3.2) which is quadratic in the derivatives of the scalars can be identified with the action of the $\mathbb{CP}^n$ sigma model described in the previous section. In the above DBI action we have set the Born-Infeld field to zero, which is a consistent truncation.

As in the case of the $\mathbb{CP}^n$ sigma models in the previous section, we shall seek static solutions of this system. The energy[2] of such solutions is

$$E_{DBI} = \int d^2x \sqrt{\det(\tilde{\gamma}_{ij})}\,, \tag{3.3}$$

where $\tilde{\gamma}_{ij}$ is the pull back of the spatial part of the metric (3.1). The solutions that minimize the energy are minimal surfaces in $\mathbb{R}^2 \times \mathbb{CP}^n$. Since $\mathbb{R}^2 \times \mathbb{CP}^n$ is a Kähler manifold, it is well known that there is a bound for the energy which is described by a Kähler calibration. In particular

$$E_{DBI} \geq 2\pi |Q|\,, \tag{3.4}$$

where Q is a topological charge induced by pulling back the Kähler form

$$\omega = dx^1 \wedge dx^2 + \omega_K \tag{3.5}$$

onto the embedded submanifold, and ω_K is the Kähler form on $\mathbb{CP}^n$. The configurations that saturate the bound are holomorphic curves in $\mathbb{R}^2 \times \mathbb{CP}^n$. So we find that the configurations that saturate the energy bound of the sigma model also saturate the calibration bound for the DBI solitons.

In terms of the parameterization, as a unit length column vector U, introduced in the previous section for the sigma model, the DBI lagrangean becomes

$$\mathcal{L}_{DBI} = \sqrt{-\det(\eta_{\mu\nu} + (D_\mu U)^\dagger (D_\nu U) + (D_\nu U)^\dagger (D_\mu U))} - 1 \tag{3.6}$$

and the static energy is

$$E_{DBI} = \int d^2x \sqrt{\det(\eta_{ij} + (D_i U)^\dagger (D_j U) + (D_j U)^\dagger (D_i U))} - 1\,, \tag{3.7}$$

where we have now subtracted off the vacuum energy.

[2]This energy functional arises naturally in the calibration bound and includes the vaccum energy.

In this formulation it is an easy exercise to verify the energy bound (3.4), where E_{DBI} and Q are given by (3.7) and (2.8) respectively, and to show that the sigma model solitons, given by (2.10) with f a holomorphic (or anti-holomorphic) function, saturate this bound.

4. Non-BPS DBI solitons

We shall now find that, remarkably, the non-BPS sigma model solitons are also solutions of the DBI field equations. To prove this we begin by computing the DBI field equations for static configurations that follow from the variation of (3.6). After a little algebra we find that the static field equations are

$$
\begin{aligned}
&D_i\frac{(D_iU)[1+2(D_jU)^\dagger(D_jU)]-(D_jU)[(D_iU)^\dagger(D_jU)+(D_jU)^\dagger(D_iU)]}{\sqrt{1+2(D_lU)^\dagger(D_lU)[1+(D_kU)^\dagger(D_kU)]-(D_lU)^\dagger(D_kU)[(D_lU)^\dagger(D_kU)+(D_kU)^\dagger(D_lU)]}}\\
&=-\frac{U\{(D_iU)^\dagger(D_iU)[1+2(D_jU)^\dagger(D_jU)]-(D_iU)^\dagger(D_jU)[(D_iU)^\dagger(D_jU)+(D_jU)^\dagger(D_iU)]\}}{\sqrt{1+2(D_lU)^\dagger(D_lU)[1+(D_kU)^\dagger(D_kU)]-(D_lU)^\dagger(D_kU)[(D_lU)^\dagger(D_kU)+(D_kU)^\dagger(D_lU)]}}.
\end{aligned}
\tag{4.1}
$$

It is convenient to introduce the notation $T_{ij}\equiv(D_iU)^\dagger(D_jU)$ and rewrite equation (4.1) in the form

$$
\begin{aligned}
&D_1\left\{\frac{(D_1U)(1+2T_{22})-(D_2U)(T_{12}+T_{21})}{\sqrt{1+2T_{11}+2T_{22}+4T_{11}T_{22}-(T_{12}+T_{21})^2}}\right\}+\\
&+D_2\left\{\frac{(D_2U)(1+2T_{11})-(D_1U)(T_{12}+T_{21})}{\sqrt{1+2T_{11}+2T_{22}+4T_{11}T_{22}-(T_{12}+T_{21})^2}}\right\}+\\
&\qquad+\frac{U[T_{11}+T_{22}+4T_{11}T_{22}-(T_{12}+T_{21})^2]}{\sqrt{1+2T_{11}+2T_{22}+4T_{11}T_{22}-(T_{12}+T_{21})^2}}=0\,.
\end{aligned}
\tag{4.2}
$$

Next we observe that if we impose the following constraints

$$T_{11}=T_{22}\,,\qquad T_{12}=-T_{21} \tag{4.3}$$

then equations (4.2) can be simplified to

$$D_1D_1U+D_2D_2U+U(T_{11}+T_{22})=0\,, \tag{4.4}$$

which are exactly the sigma model field equations for static configurations (2.7). Thus if we can find static solutions of the sigma model equations which also satisfy the constraints (4.3) then they will also be static solutions of the DBI equations. The BPS solitons solve the equations (2.9) and so automatically obey the constraints (4.3) (and have the additional property that $T_{12}=\pm iT_{11}$). We shall now show that the sigma model non-BPS solitons also satisfy the constraints.

In terms of the complex variable $z = x_1 + ix_2$ the pair of real equations (4.3) can be written as the single complex equation

$$(D_z U)^\dagger (D_{\bar{z}} U) = 0\,. \tag{4.5}$$

From the definition of P_+, given by (2.11), the following properties may be proved (see eg. [11]) when f is a holomorphic vector,

$$\begin{aligned} (P_+^k f)^\dagger\, P_+^l f &= 0\,, \qquad k \neq l \\ \partial_{\bar{z}} \left(P_+^k f\right) &= -P_+^{k-1} f \frac{|P_+^k f|^2}{|P_+^{k-1} f|^2}\,, \\ \partial_z \left(\frac{P_+^{k-1} f}{|P_+^{k-1} f|^2}\right) &= \frac{P_+^k f}{|P_+^{k-1} f|^2}\,. \end{aligned} \tag{4.6}$$

Using these properties it is elementary to show that if U is a non-BPS solution given by (2.12) then

$$D_z U = \frac{P_+^{k+1} f}{|P_+^k f|}\,, \qquad D_{\bar{z}} U = -P_+^{k-1} f \frac{|P_+^k f|}{|P_+^{k-1} f|^2}\,. \tag{4.7}$$

Hence equation (4.5) follows immediately from the orthogonality property (4.6). This completes the proof that the non-BPS solitons are also solutions of the DBI model.

These non-BPS solutions have the interpretation of soliton/anti-soliton states just as in the sigma model. As in the sigma model [11] these solutions are therefore expected to be unstable configurations. The global properties of such solutions as submanifolds of $\mathbb{CP}^n$ have been investigated in [12].

The reader may wonder if it is possible to see why the non-BPS solitons solve the DBI equations directly from the lagrangean. Of course, in general it is inconsistent to substitute constraints in the lagrangean since the critical points of the constrained system are generally not critical points of the full theory. For BPS solitons this is not a concern because they saturate the energy bound and so are automatically critical points. It is therefore enough to show that the DBI lagrangean reduces to the sigma model lagrangean when the BPS constraints are imposed. However, non-BPS solutions require a little more thought.

As we want to investigate when the DBI lagrangean (or equivalently energy density since we are dealing with the static sector) reduces to that of the sigma model then we compute that

$$\begin{aligned} &(1+\mathcal{E}_\sigma)^2 - (1+\mathcal{E}_{DBI})^2 = \\ &\quad = (1 + T_{11} + T_{22})^2 - (1 + 2T_{11} + 2T_{22} + 4T_{11}T_{22} - (T_{12} + T_{21})^2) \\ &\quad = (T_{11} - T_{22})^2 + (T_{12} + T_{21})^2\,. \end{aligned} \tag{4.8}$$

In the final line we recognize the constraints (4.3) and there are two crucial points to be noted. The DBI energy density is identical to that of the sigma model if and only if the constraints (4.3) hold. The first point is that these constraints are weaker than the BPS equations and have more solutions, in particular we have shown that the non-BPS solitons satisfy the constraints. The second point is that the above expression is quadratic in the constraints, so its variation is proportional to the constraints and vanishes after their imposition. This explains why the critical points of the constrained system, which is the sigma model, are also critical points of the full DBI theory.

5. Sigma model vs DBI dynamics

As we have seen, static sigma model BPS solitons also solving the DBI field equations. It is therefore natural to compare the dynamics of slowly moving BPS solitons in the sigma model and in the DBI model, using the moduli space approximation [13]. Naively it might be expected that the low energy dynamics of BPS solutions in the two theories will be the same since they have the same low energy limit. However, the agreement of the two lagrangeans is to quadratic order in all the derivatives of the fields i.e. space and time, whereas the moduli space approximation assumes that time derivatives are small but there is no truncation of the spatial derivatives. Thus it is not clear whether the dynamics of the DBI solitons will match that of the sigma model solitons. The moduli space of sigma model solitons is a Kähler manifold [14].

The moduli space approximation truncates the full field dynamics to motion on the BPS soliton moduli space. Applying the BPS soliton equation (2.9) the DBI lagrangean (3.2) can be written as

$$\mathcal{L}_{DBI} = \sqrt{(1+2T_{11})(1+2T_{11}-2T_{00}-4(T_{11}T_{00}-T_{01}T_{10}))} - 1\,, \tag{5.1}$$

where the notation is as in section four. Expanding out the square root and neglecting terms which are higher order than quadratic in the *time* derivatives, we obtain, after the integration over space

$$L_{DBI} = 2\pi|Q| - \int d^2x(T_{00} + 2(T_{11}T_{00} - T_{01}T_{10}))\,. \tag{5.2}$$

The first term is just the potential energy of a charge Q soliton and the remaining term defines a metric on the Q-soliton moduli space, with respect to which the dynamics is approximated by geodesic motion. The first term in the integrand is the kinetic energy of the sigma model, thus the metric on the moduli space of DBI solitons will be equal to that of the sigma model solitons if and only if the term (which we now write out in full)

$$K \equiv 2\int \{(D_1U)^\dagger(D_1U)(D_0U)^\dagger(D_0U) - (D_0U)^\dagger(D_1U)(D_1U)^\dagger(D_0U)\}\, d^2x \tag{5.3}$$

vanishes or is equal to a total time derivative.

For the $\mathbb{CP}^1$ model this term is indeed zero. The easiest way to see this is to choose the parameterization (which can be done without loss of generality using the local U(1) symmetry)

$$U = \frac{1}{\sqrt{1+|w|^2}} \begin{pmatrix} 1 \\ w \end{pmatrix} \tag{5.4}$$

in terms of which it is easily found that

$$D_\mu U = \frac{\partial_\mu w}{(1+|w|^2)^{3/2}} \begin{pmatrix} -\bar{w} \\ 1 \end{pmatrix}. \tag{5.5}$$

Thus $D_0 U$ is proportional to $D_1 U$ and hence the integrand in (5.3) is identically zero. However, this cancellation is a unique property of the $\mathbb{CP}^1$ case and derives from the fact that it has only one (complex) field.

To demonstrate that in the $\mathbb{CP}^n$ model with $n > 1$ the term (5.3) can be non-zero it is enough to consider the $\mathbb{CP}^2$ case (since $\mathbb{CP}^2$ is a totally geodesic submanifold of $\mathbb{CP}^n$ for $n > 2$). Writing the $\mathbb{CP}^2$ field as

$$U = \frac{1}{\sqrt{1+|w_1|^2+|w_2|^2}} \begin{pmatrix} 1 \\ w_1 \\ w_2 \end{pmatrix}, \tag{5.6}$$

then

$$K = \int \frac{2|\partial_0 w_1 \partial_z w_2 - \partial_0 w_2 \partial_z w_1|^2}{(1+|w_1|^2+|w_2|^2)^3} d^2x\,. \tag{5.7}$$

If either w_1 or w_2 are identically zero then this term vanishes, corresponding to the embedding of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$, but generally (5.7) is non-zero. Finally, we need to check that this term is not a total time derivative, otherwise it would not contribute to the geodesic equations. The easiest way to show this is by a simple example.

An axially symmetric $\mathbb{CP}^2$ soliton of charge Q has the form

$$w_1 = \alpha z^Q - \beta\,, \qquad w_2 = \alpha z^Q + \beta\,, \tag{5.8}$$

where α and β are complex parameters related to the size and shape of the soliton [15]. Taking α and β to be time dependent the integrand in (5.7) is axially symmetric and the integration is elementary to arrive at

$$K = \frac{2\pi Q |\partial_0 \beta|^2}{(1+2|\beta|^2)^2}\,, \tag{5.9}$$

which is clearly not a total time derivative.

One expects that in a supersymmetric extension the $\mathbb{CP}^n$ DBI solitons for $n > 1$ break more supersymmetry[3] than those in $\mathbb{CP}^1$. Because of this, the above observation regarding the low energy dynamics of solitons, is related to a similar observation

[3]We remark that the corresponding sigma model solitons always break half of the supersymmetry of the $N = 2$ $(2+1)$-dimensional sigma model.

JHEP09(1999)016

in [16] that the small perturbations around supersymmetric solitons that preserve less than 1/4 of the maximal spacetime supersymmetry in the Maxwell Theory-Sigma Model approximation of the DBI action do not solve the perturbed DBI field equations. However if the solutions preserve 1/4 of spacetime supersymmetry, then the perturbations of the Maxwell Theory-Sigma Model approximation also solve the perturbed DBI field equations. This indicates that for BPS solitons which preserve enough supersymmetry, supersymmetry protects the sigma model soliton moduli metric from higher derivative corrections.

In summary we have shown that for $\mathbb{CP}^1$ the slow motion dynamics of DBI solitons is identical to that of the sigma model solitons, but for $n > 1$ the dynamics of slowly moving $\mathbb{CP}^n$ DBI solitons is different from that of the sigma model solitons. This is despite the fact that the low energy truncation of the DBI lagrangean gives the sigma model one and that the BPS solutions of the two systems are the same.

6. Non-BPS Born-Infeld field configurations

The investigation of the relation between BPS and non-BPS solutions of DBI theory and those of sigma models, that we have described, can be extended to include the relation between BPS and non-BPS solutions of DBI theory and those of Yang-Mills coupled to matter systems. Recently SU(n) monopole solutions have been constructed which solve the second order Yang-Mills-Higgs equations but are not solutions of the first order Bogomolny equations. Not only are these solutions three-dimensional analogues of the non-BPS $\mathbb{CP}^{n-1}$ sigma model solutions but in fact the sigma model solutions are used explicitly to obtain the monopoles [17]. An obvious candidate in the search for non-BPS DBI solitons is therefore to consider a system of n parallel D3-branes in type IIB string theory, since its low energy truncation reproduces the Yang-Mills-Higgs lagrangean. However, we shall make a simple observation that suggests that the non-Bogomolny Yang-Mills-Higgs monopoles will not be solutions of the DBI equations in this case. This in fact may not be a surprise. For example, in [18] it has been observed that even some BPS solutions of Yang-Mills theory do not solve the Born-Infeld field equations.

Although the effective action of a single D3-brane in type IIB string theory is described by an abelian DBI lagrangean a system of n parallel D3-branes is expected to be described by a U(n) DBI theory, the most promising of which is the proposal of Tseytlin [19]. In the static case, with one active adjoint scalar, the energy density of the D3-brane worldvolume theory can be conveniently written as [20]

$$\mathcal{E}_{BI} = \mathrm{STr}(\sqrt{\det(\delta_{ab} + F_{ab})} - 1)\,, \tag{6.1}$$

where a, b go from 1 to 4 and a dimensional reduction is performed in the x_4 direction with the usual identification of the scalar field as $\Phi = A_4$. The gauge group is SU(n)

(an overall U(1) factor decouples as the centre of mass) and STr denotes the trace over gauge indices of the weighted sum over all permutations of the non-commutative products [19]. This is required in order to make sense of the ordering ambiguities involved in computing the determinant.

Expanding (6.1) to quadratic order in the fields reproduces the static Yang-Mills-Higgs energy density

$$\mathcal{E}_{YMH} = \mathrm{Tr}\left(\frac{1}{2}D_i\Phi D_i\Phi + \frac{1}{4}F_{jk}F_{jk}\right), \tag{6.2}$$

where $i = 1, 2, 3$. The BPS monopole solutions of (6.2), which satisfy the Bogomolny equation

$$D_i\Phi = \pm\frac{1}{2}\epsilon_{ijk}F_{jk} \tag{6.3}$$

are also solutions of the Born-Infeld theory (6.1), which can be shown by noting that the two energies (6.1) and (6.2) are equal upon substitution of the Bogomolny equation [20, 21]. The charge k monopole solution describes k D-strings stretched between the n D3-branes and this can be seen explicitly by graphing the eigenvalues of the scalar field Φ over $\mathbb{R}^3$ [20].

In order to determine whether the non-Bogomolny monopole solutions [17] of (6.2) are solutions of the Born-Infeld theory we follow the procedure given in (4.8) for the sigma model case, and compute when the Yang-Mills-Higgs and Dirac-Born-Infeld energies agree. Ignoring the trace operation and treating the matrices as if they were abelian (which can be justified using the symmetrized trace [20, 21]) we compute that

$$\begin{aligned} 2[(1+\mathcal{E}_{YMH})^2 - (1+\mathcal{E}_{BI})^2] = {} & (D_2\Phi D_3\Phi + F_{12}F_{13})^2 + (D_1\Phi D_3\Phi + F_{21}F_{23})^2 + \\ & + (D_1\Phi D_2\Phi + F_{31}F_{32})^2 + (D_2\Phi F_{12} + D_3\Phi F_{13})^2 + \\ & + (D_1\Phi F_{21} + D_3\Phi F_{23})^2 + (D_1\Phi F_{31} + D_2\Phi F_{32})^2 + \\ & + \frac{1}{2}((D_1\Phi)^2 - (F_{23})^2)^2 + \frac{1}{2}((D_2\Phi)^2 - (F_{13})^2)^2 + \\ & + \frac{1}{2}((D_3\Phi)^2 - (F_{12})^2)^2 . \end{aligned} \tag{6.4}$$

As in the sigma model case, (4.8), we find that the result can again be written as a sum of squares, but this time there is an important difference in that the constraints (under which the two energies agree) now contain the Bogomolny equations (6.3) explicitly (see the last three terms). Thus in this case, in contrast to the sigma model example, the constraints are solved only by the Bogomolny monopoles. Although this does not prove that the non-Bogomolny monopoles do not solve the Born-Infeld theory, it strongly suggests that they do not and certainly shows that the feature of non-BPS solitons in the sigma model does not apply in the same way to this example.

7. Concluding remarks

We have shown that the $\mathbb{CP}^n$ sigma model BPS solitons also solve the field equations of a Dirac-Born-Infeld action. Furthermore, we have shown that certain non-BPS $\mathbb{CP}^n$ sigma model solutions, which correspond to soliton/anti-soliton configurations, are also solutions of the Dirac-Born-Infeld action. We have also investigated the dynamics of the $\mathbb{CP}^n$ DBI solitons and found that they do not coincide with the sigma model dynamics unless $n = 1$. Finally, we explored the possibility of similar non-BPS solutions in Yang-Mills theories.

The possible D-brane interpretation of our results remains an open problem. Although it is expected that some sigma model solitons can be embedded in a brane theory there are several restrictions on the DBI action, such as kappa symmetry. In particular, this implies that the sigma model target space should be a solution of the supergravity field equations. Since there are no supergravity solutions which are topologically $\mathbb{CP}^n$ then it is not obvious that any of our solutions have a D-brane interpretation. However, there are several possibilities which may admit a D-brane interpretation. For example, the near horizon geometry of the M-2-brane is $AdS_4 \times S^7$ and since S^7 is a circle bundle over $\mathbb{CP}^3$ then by a Kaluza-Klein reduction along the circle fibre it is possible to obtain a background which includes $\mathbb{CP}^3$ [22]. Our sigma model solitons can be embedded as solutions in this case but unfortunately this requires a singular embedding and so there is no obvious D-brane interpretation for this example.

Acknowledgments

Many thanks to Gary Gibbons, Nick Manton and Wojtek Zakrzewski for useful discussions. PMS acknowledges the EPSRC for an Advanced Fellowship and the grant GR/L88320. GP is supported by a Royal Society Research Fellowship.

References

[1] J.H. Hughes, J. Liu, J. Polchinski, *Supermembranes*, *Phys. Lett.* **B 180** (1986) 370;
R.G. Leigh, *Dirac-Born-Infeld action from Dirichlet sigma model*, *Mod. Phys. Lett.* **A 4** (1989) 2767.
E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and 11 dimensional supergravity*, *Phys. Lett.* **B 189** (1987) 75.
P.S. Howe and E. Sezgin, $D = 11$, $P = 5$, *Phys. Lett.* **B 394** (1997) 62 [hep-th/9611008].

[2] G. Papadopoulos and P.K. Townsend, *Intersecting M-branes*, *Phys. Lett.* **B 380** (1996) 273 [hep-th/9603087].

[3] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[4] K. Becker, M. Becker and A. Strominger, *Fivebranes, membranes and non-perturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158].

[5] J. Gutowski and G. Papadopoulos, *AdS calibrations*, hep-th/9902034.

[6] E. Bergshoeff and P.K. Townsend, *Solitons on the supermembrane*, *J. High Energy Phys.* **05** (1999) 021 [hep-th/9904020].

[7] J. Gutowski, G. Papadopoulos and P.K. Townsend, *Supersymmetry and generalized calibrations*, hep-th/9905156.

[8] R.S. Ward, *Slowly-moving lumps in the* $\mathbb{CP}^1$ *model in (2+1) dimensions*, *Phys. Lett.* **B 158** (1985) 424.
R. Leese, *Low-energy scattering of solitons in the* $\mathbb{CP}^1$ *model*, *Nucl. Phys.* **B 344** (1990) 33.

[9] B. Piette, P.M. Sutcliffe and W.J. Zakrzewski, *Soliton antisoliton scattering in* $2+1$ *dimensions*, *Int. J. Mod. Phys.* **C 3** (1992) 637.

[10] A.M. Din and W.J. Zakrzewski, *General classical solutions of the* $\mathbb{CP}^{n-1}$ *model*, *Nucl. Phys.* **B 174** (1980) 397.

[11] W. J. Zakrzewski, *Low dimensional sigma models*, IOP, 1989.

[12] J. Eells and J.C. Wood, *Harmonic maps from surfaces to complex projective spaces*, *Advances in Mathematics* **49** (1983) 217.

[13] N.S. Manton, *A remark on scattering of BPS monopoles*, *Phys. Lett.* **B 110** (1982) 54.

[14] P.J. Ruback, *Sigma model solitons and their moduli space metric*, *Comm. Math. Phys.* **116** (1988) 645.

[15] B.M.A.G. Piette, M.S.S. Rashid and W.J. Zakrzewski, *Soliton scattering in the* $\mathbb{CP}^2$ *model*, *Nonlinearity* **6** (1993) 1077.

[16] J. Gutowski and G. Papadopoulos, *The moduli spaces of worldvolume solitons*, *Phys. Lett.* **B 432** (1998) 97 [hep-th/9802186]; *The dynamics of D-3-brane dyons and toric hyper-Kähler manifolds*, *Nucl. Phys.* **B 551** (1999) 650 [hep-th/9811207].

[17] T. Ioannidou and P.M. Sutcliffe, *Non-Bogomolny* $\mathrm{SU}(N)$ *BPS monopoles*, hep-th/9905169.

[18] G.W. Gibbons and G. Papadopoulos, *Calibrations and intersecting branes*, *Comm. Math. Phys.* **202** (1999) 593 [hep-th/9803163].

[19] A.A. Tseytlin, *On non-abelian generalization of Born-Infeld action in string theory*, *Nucl. Phys.* **B 501** (1997) 41 [hep-th/9701125].

[20] A. Hashimoto, *The shape of branes pulled by strings*, *Phys. Rev.* **D 57** (1998) 6441 [hep-th/9711097].

[21] D. Brecher, *BPS states of the non-abelian Born-Infeld actions*, *Phys. Lett.* **B 442** (1998) 117 [hep-th/9804180].

[22] M.J. Duff, H. Lü and C.N. Pope, $AdS_5 \times S^5$ *untwisted*, *Nucl. Phys.* **B 532** (1998) 181 [hep-th/9803061];
M.J. Duff, *Anti-de Sitter space, branes, singletons, superconformal field theories and all that*, *Int. J. Mod. Phys.* **A 14** (1999) 815 [hep-th/9808100].